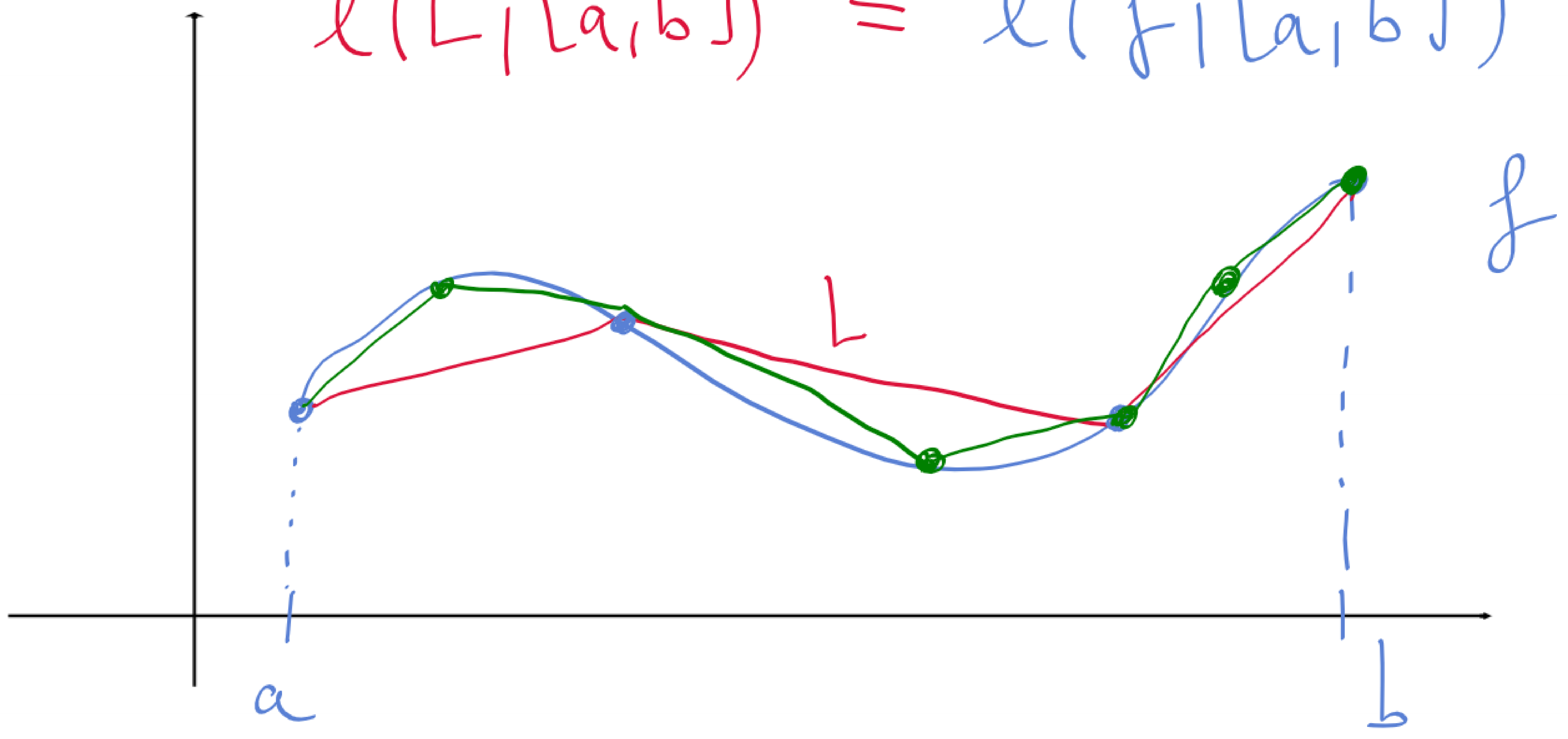
Odvození vzorce pro délku grafu funkce

$$\ell(L,[a,b]) \le \ell(f,[a,b])$$



Chceme spočítat délku grafu f na $[a,b]$, označíme $\ell(f,[a,b])$.

Poznámka: • I pro spojitou funkci na $[a,b]$ se může stát, že $\ell(f,[a,b]) = \infty$.

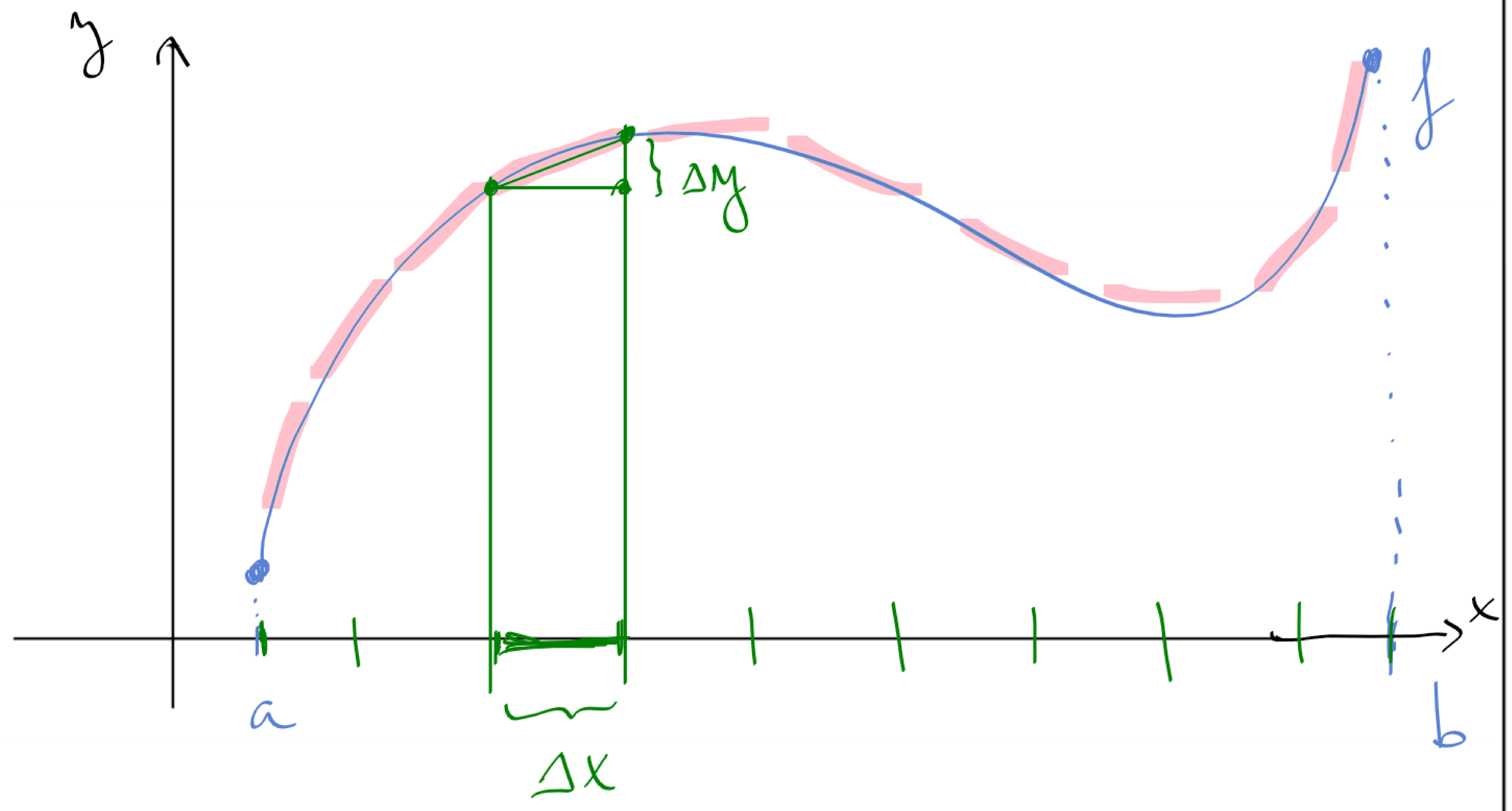
• dokonce se to může stát pro diferencovatelnou funkci f.

• my budeme předpokládat, že

$$f \in C^1([a,b]) := \{ f : f \text{ má spojitou derivaci na } [a,b] \}$$

Definice:

$$\ell(f,[a,b]) := \sup\{ \ell(L,[a,b]) : L \text{ je lomenice na } [a,b], \text{ že koncové body jednotlivých úseček jsou body grafu funkce } f \}$$



$$\sum \sqrt{\Delta x^2 + \Delta y^2} = \sum \underbrace{\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}}_{\sqrt{1+(f')^2}} \cdot \Delta x$$

$$\xrightarrow{\Delta x \to 0} \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

Příklady: $f(x) = \sqrt{1-x^2}$ na $[0,1]$.

$$l(f,[0,1]) = \int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\, dx =$$

$$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$(f'(x))^2 = \frac{x^2}{1-x^2}$$

$$\int_0^1 \sqrt{\frac{1}{1-x^2}}\, dx$$

ALT. ..

$$= \left| \begin{array}{l} x = \sin t \\ dx = \cos t\, dt \end{array} \right| = \int_0^{\pi/2} \sqrt{\frac{1}{1-\sin^2 t}} \cos t\, dt =$$

x	0	1
t	0	$\frac{\pi}{2}$

$\cos t \geq 0$, $t \in [0, \frac{\pi}{2}]$

$$= \int_0^{\pi/2} \sqrt{\frac{1}{\cos^2 t}} \cdot \cos t\, dt = \int_0^{\pi/2} \frac{1}{|\cos t|} \cdot \cos t\, dt = \int_0^{\pi/2} 1\, dt = \pi/2$$

$$\int_0^1 \sqrt{\frac{1}{\underbrace{1-x^2}_{(1-x)(1+x)}}}\,dx = \int_0^1 \sqrt{\frac{1-x}{1+x}\cdot\frac{1}{(1-x)^2}}\,dx =$$

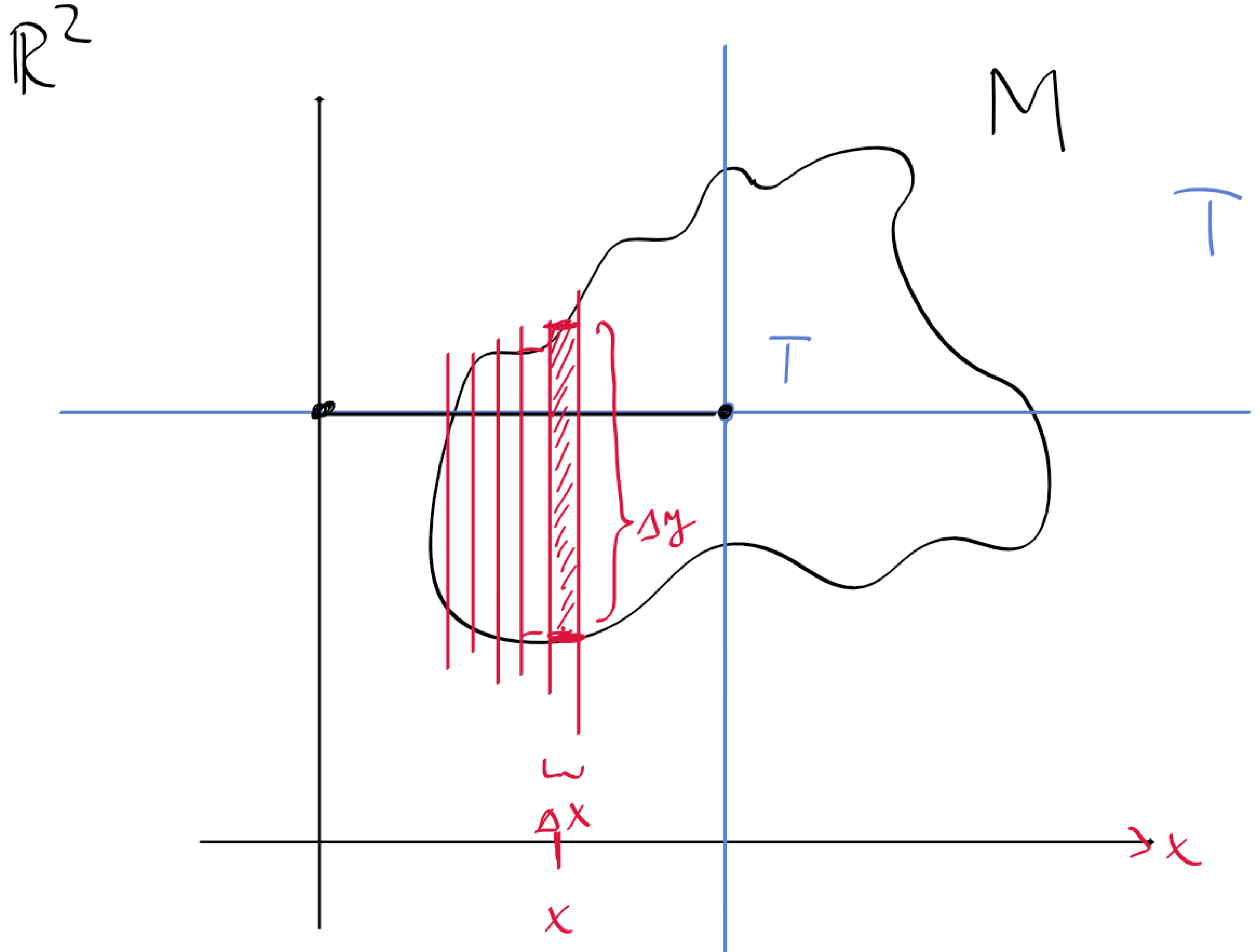
$$= \int_0^1 \sqrt{\frac{1-x}{1+x}}\cdot\frac{1}{1-x}\,dx = \left|\; y = \sqrt{\frac{1-x}{1+x}} \quad \dots \right.$$

nede k cíli

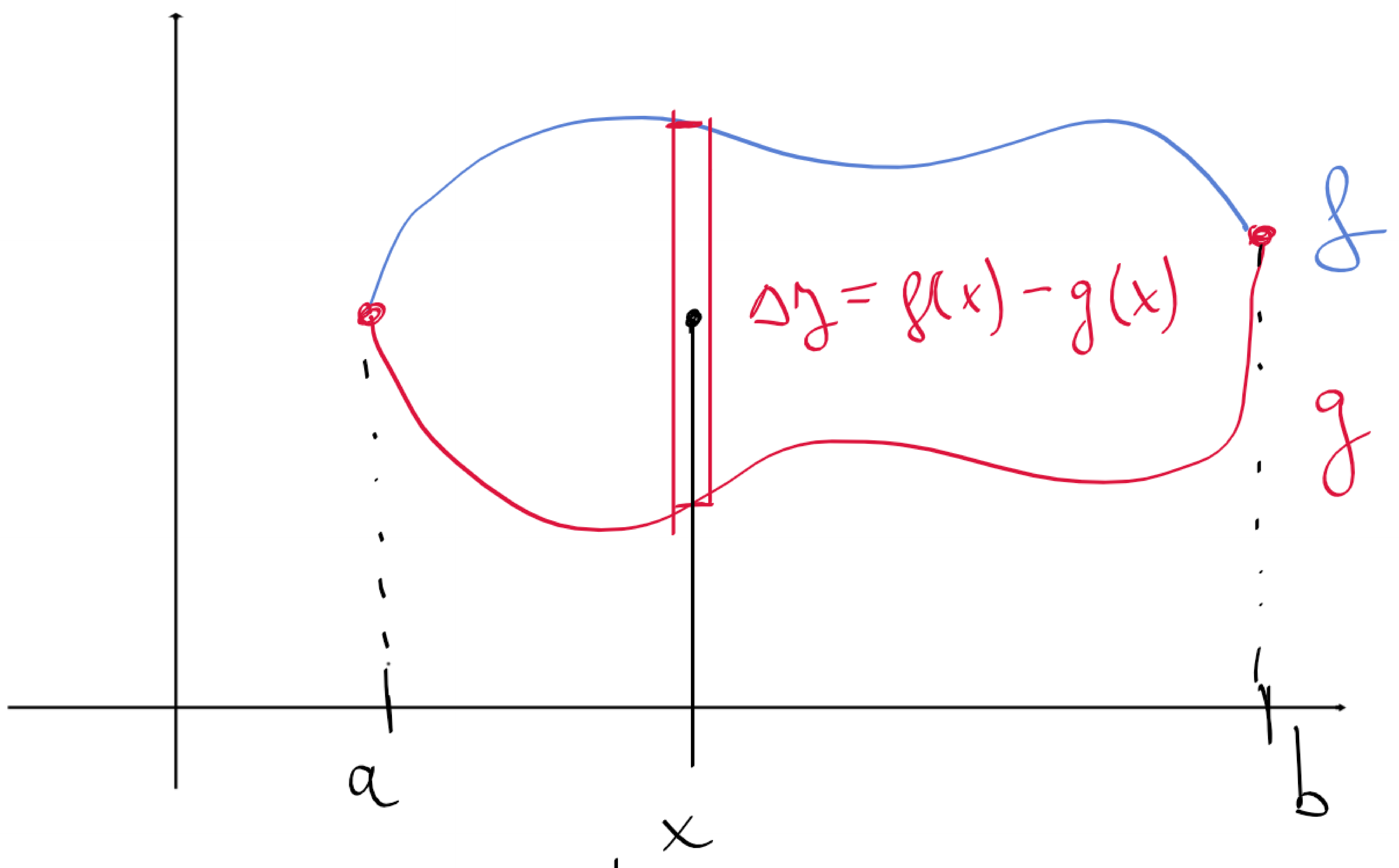
$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \Big[\arcsin x\Big]_0^1 =$$

$$= \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0$$

TĚŽIŠTĚ ROVINNÉHO ÚTVARU



$T = [T_x, T_y]$



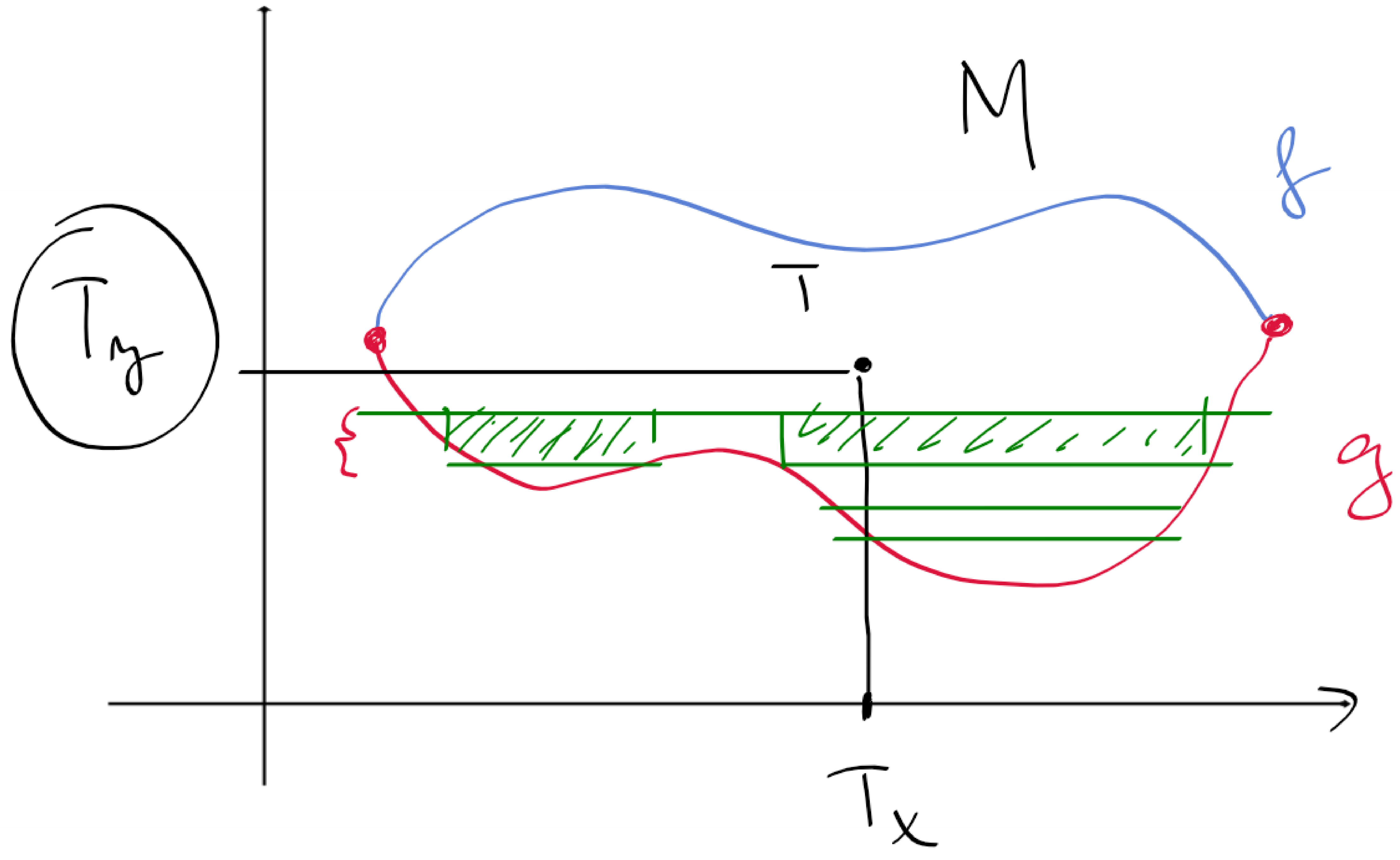
Chceme: $T_x \cdot S_M = \sum x \cdot \Delta x \cdot \Delta y$

$[\Delta x \to 0]$

$$T_x \cdot S_M = \int_a^b x \cdot \underbrace{(f(x) - g(x))}_{\Delta y} \, dx$$

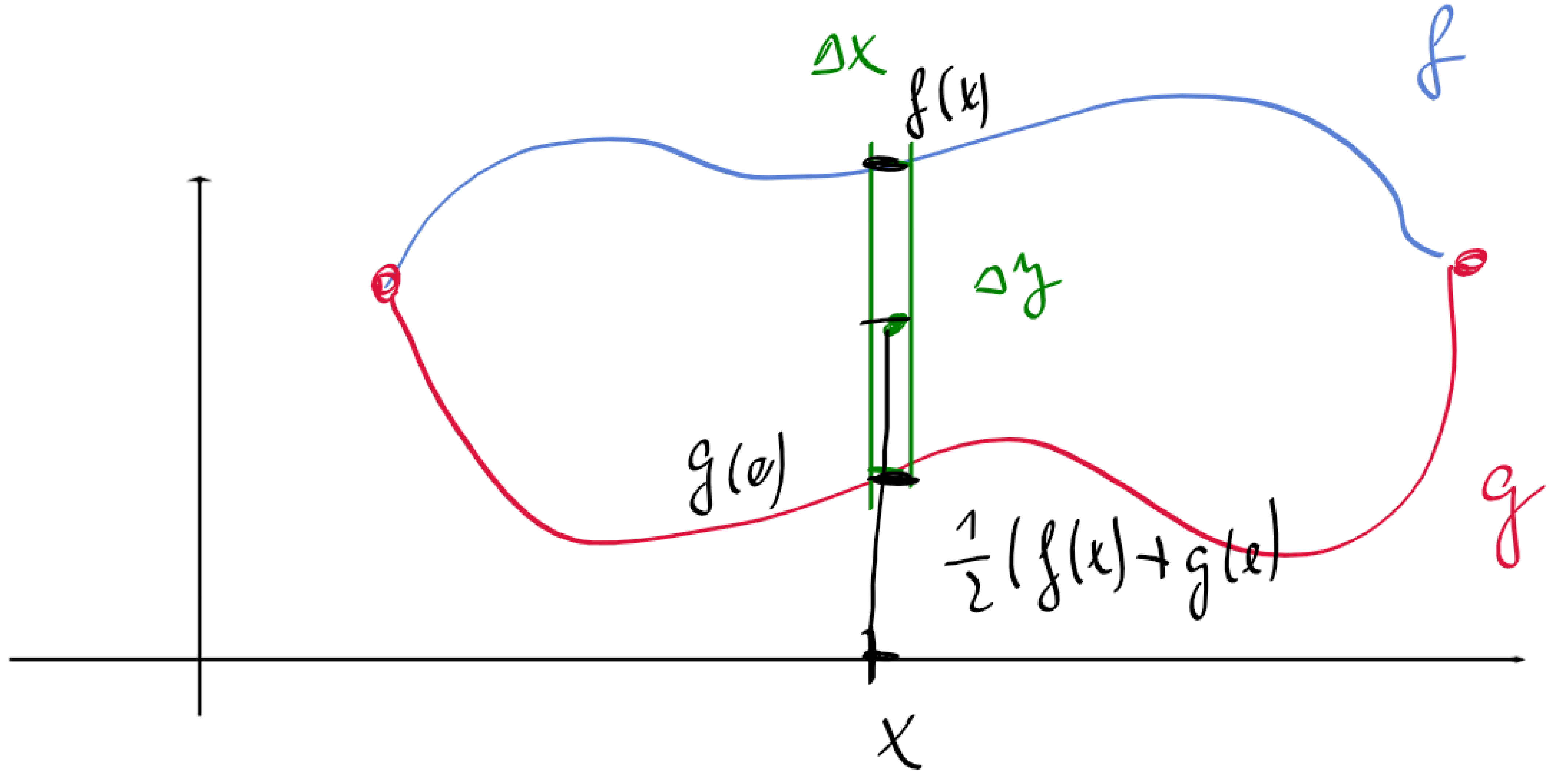
$$T_x = \frac{\int_a^b x \cdot (f(x) - g(x)) \, dx}{\int_a^b (f(x) - g(x)) \, dx}$$

T_y :



nemusí být konvexní ⇒ vodorovné pásečky

nemusí být „celistvé"



$$T_y \cdot S_M = \sum \Delta x \cdot \Delta y \cdot \frac{1}{2}\left(f(x)+g(x)\right) \cancel{=}$$

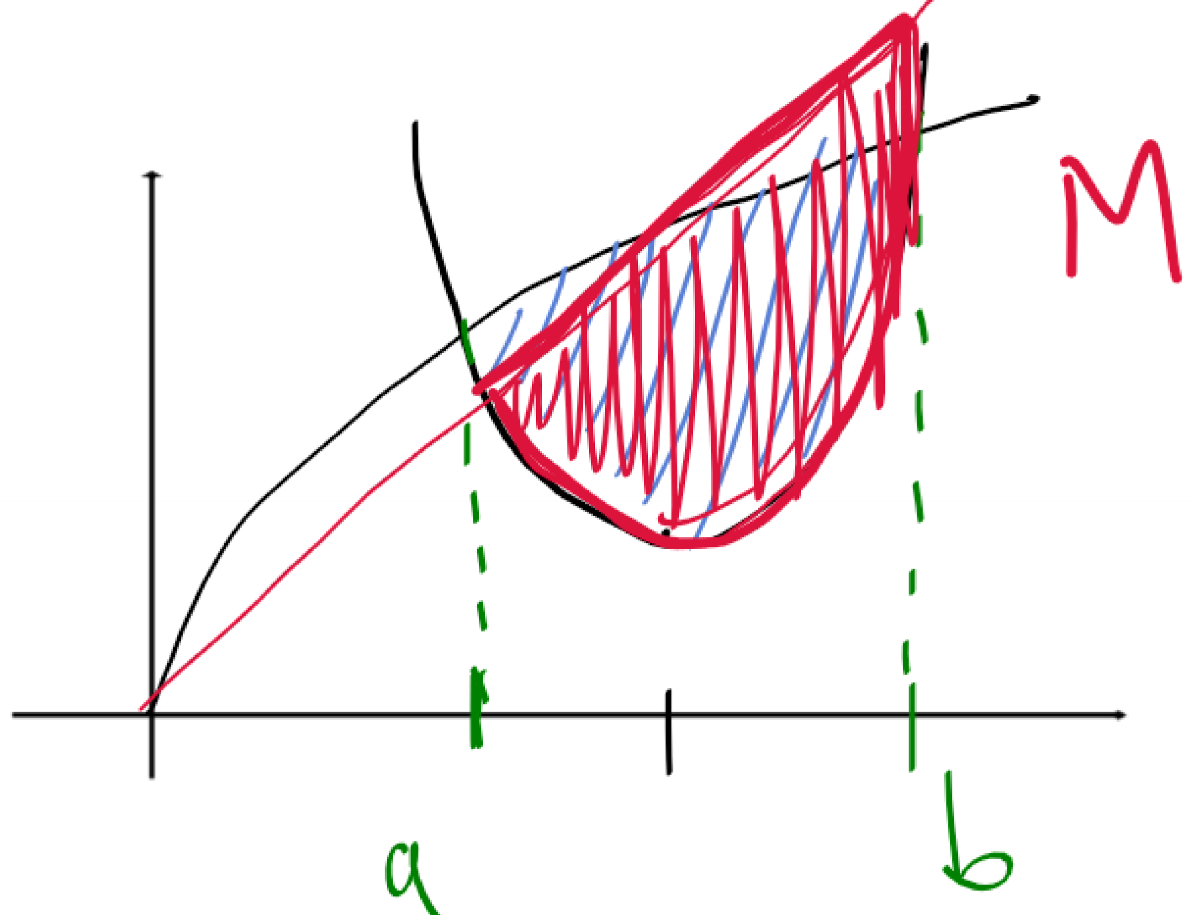
$$\Delta x \to 0 \qquad \left(\Delta x \rightsquigarrow dx,\ \Delta y = f(x)-g(x)\right)$$

$$\to \int_a^b \left(f(x)-g(x)\right) \cdot \frac{1}{2}\left(f(x)+g(x)\right) dx$$

$$T_y = \frac{\frac{1}{2}\int_a^b \left(f^2(x)-g^2(x)\right) dx}{\int_a^b \left(f(x)-g(x)\right) dx}$$

Příklad: $f(x) = x$

$g(x) = \frac{1}{2} + (x-1)^2$



$a = \frac{3-\sqrt{3}}{2}$ $\quad 1 \quad$ $b = \frac{3+\sqrt{3}}{2}$

$\parallel$

$0{,}633\ldots$

$x = \frac{1}{2} + (x-1)^2$

$x = \frac{1}{2} + x^2 - 2x + 1$

$x^2 - 3x + \frac{3}{2} = 0$

$a, b = \frac{3 \pm \sqrt{9-6}}{2} = \frac{3 \pm \sqrt{3}}{2}$

$$S_M = \int_{\frac{3-\sqrt{3}}{2}}^{\frac{3+\sqrt{3}}{2}} \left(x - \left(\frac{1}{2} + (x-1)^2\right)\right) dx =$$

$$= \left[\frac{x^2}{2} - \frac{1}{2}x - \frac{1}{3}(x-1)^3\right]_{\frac{3-\sqrt{3}}{2}}^{\frac{3+\sqrt{3}}{2}} = \ldots$$

$$T_x \cdot S_M = \int_a^b x\left(x - \frac{1}{2} - \underbrace{(x-1)^2}_{-x^2+2x-1}\right) dx =$$

$$= \int_a^b \left(x^2 - \frac{1}{2}x - x^3 + 2x^2 - x\right) dx =$$

$$= \int_a^b \left(-x^3 + 3x^2 - \frac{3}{2}x\right) =$$

$$= \left[-\frac{x^4}{4} + x^3 - \frac{3}{4}x^2\right]_{\frac{3-\sqrt{3}}{2}}^{\frac{3+\sqrt{3}}{2}}$$

$T_x = \frac{3}{2}$ … vyjde po správném dosazení

$$T_y \cdot S_M = \frac{1}{2}\int_a^b \left(f^2(x) - g^2(x)\right) dx =$$

$$= \frac{1}{2}\int_a^b \left(x^2 - \left(\frac{1}{2} + (x-1)^2\right)^2\right) dx =$$

$$= \frac{1}{2}\int_a^b \left(x^2 - \frac{1}{4} - (x-1)^2 - (x-1)^4\right) dx =$$

$$= \frac{1}{2}\int_a^b \left(\cancel{x^2} - \cancel{\frac{1}{4}} - \cancel{x^2} + \cancel{2x} - \cancel{1} - \left(x^4 - 4x^3 + 6x^2 - \cancel{4x} + \cancel{1}\right)\right) dx$$

$$= \frac{1}{2}\int_a^b \left(-9/4 + 6x - 6x^2 + 4x^3 - x^4\right) dx =$$

$$= \frac{1}{2}\left[-\frac{9}{4}x + 3x^2 - 2x^3 + x^4 - \frac{x^5}{5}\right] = \frac{3\cdot\sqrt{3}}{5}$$

Tedy $T_y = \frac{3\sqrt{3}}{5} \Big/ S_M = \frac{3\sqrt{3}}{5} \Big/ \frac{\sqrt{3}}{2} = \frac{6}{5}$

Celkem: Těžiště T uvažované plochy je

$$T = \left[\frac{3}{2}, \frac{6}{5}\right]$$